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Time-dependent scaling relations and a cascade model of turbulence

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We study the time-dependent solutions of a nonlinear cascade model for homogeneous isotropic turbulence first introduced by Novikov & Desnyansky. The dynamical variables of the model are the turbulent kinetic energies in discrete wave-number shells of thickness one octave. The model equations contain a parameter C whose size governs the amount of energy cascaded to small wavenumbers relative to the amount cascaded to large wavenumbers. We show that the equations permit scale-similar evolution of the energy spectrum. For $0 \leqslant C \leqslant 1$ and no external force, the freely evolving energy spectrum displays the Kolmogorov $k^{-\frac{5}{3}}$ power law, and the total energy decreases in time as a power t^{-w}, where the exponent w depends on the value of C. Grid-turbulence experiments seem to favour a value of C in the range 0·3–0·6. In the presence of an external stirring force acting near a wavenumber k_0, the model predicts, in addition to the Kolmogorov $k^{-\frac{5}{3}}$ spectrum for $k > k_0$, a scale-similar flow of energy to wavenumbers $k < k_0$. This backward energy flow falls off as a power law in time, and establishes a stationary energy spectrum for $k < k_0$ which is a power law in k less steep than $k^{-\frac{5}{3}}$. We discuss the similarity of the behaviour of the model for $C > 1$ to the behaviour of turbulent fluid for a spatial dimensionality near 2. The model is shown to approach the Kovasznay and the Leith diffusion approximation equations in the limit in which the thickness of the wavenumber shells approaches zero. However, the cascade model with finite shell thicknesses appears to behave in a more physically reasonable way than the limiting differential equations.

1. Introduction

Because of the difficulty of dealing directly with the Navier–Stokes equations, numerous models have been proposed in an effort to capture some of the essential statistical features of turbulent fluid motion. Most concentrate on the evolution in time of the energy spectrum $E(k,t)$ of homogeneous isotropic turbulence. One particularly simple class of such models was introduced by Oboukhov (1971) and Desnyansky & Novikov (1974*a*). In these models a discrete set of variables $u_n(t)$, labelled by integers n, represents the energy spectrum. They are defined such that

$$\tfrac{1}{2}u_n^2(t) \equiv \int_{2^{-\frac{1}{2}}k_n}^{2^{\frac{1}{2}}k_n} E(k,t)\,dk \tag{1.1}$$

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is the energy contained in a shell in wavenumber space centred around $\mathbf{k} = 0$ with inner radius $2^{-\frac{1}{2}}k_n$ and outer radius $2^{\frac{1}{2}}k_n$, and with the wavenumbers k_n logarithmically spaced:

$$k_n \equiv 2^n k_0. \tag{1.2}$$

Desnyansky & Novikov proposed nonlinear equations of motion for these variables subject to the constraints that the equations (i) have only quadratic nonlinear terms, (ii) introduce no intrinsic length scales via the coupling coefficients, (iii) couple nearest-neighbour shells only (thereby incorporating the original Kolmogorov (1941) picture of an energy cascade) and (iv) conserve the total energy of the system

$$E(t) \equiv \sum_n \tfrac{1}{2}u_n^2(t) \tag{1.3}$$

in the absence of viscous dissipation or an external force. The most general set of equations satisfying these conditions is

$$du_n/dt = \alpha k_n[u_{n-1}^2 - 2u_n u_{n+1} - 2^{\frac{1}{3}}C(u_{n-1}u_n - 2u_{n+1}^2)] - \nu k_n^2 u_n + \mathscr{F}_n(t). \tag{1.4}$$

The parameters α and C are not fixed by the constraints. The last two terms in the equations represent the effects of viscosity and an externally imposed driving force.

Considering its simplicity, the model behaves in a remarkably sensible way. Desnyansky & Novikov (1974*a*, *b*) studied the equations with $C = 0$ and showed that behaviour corresponding to the Kolmogorov spectrum

$$E(k,t) \sim k^{-\frac{5}{3}} \tag{1.5}$$

appears both for the forced equations and with no force and appropriate initial conditions. Bell & Nelkin (1977) have shown that with an external force acting at wavenumbers near k_0, for any value of C, the equations generate a steady-state spectrum corresponding to Kolmogorov's full scaling form

$$E(k) \propto \epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}F(k/K_d) \quad (k > k_0), \tag{1.6}$$

with

$$K_d = \epsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}}, \tag{1.7}$$

where ϵ is the rate of energy dissipation by viscosity.

However, an unexpected qualitative change in the character of the scaling form occurs at the value $C = 1$. For $C < 1$, the 'universal function' $F(z)$ in (1.6) has a finite limit for $z \rightarrow 0$, so a pure $k^{-\frac{5}{3}}$ power-law spectrum persists in the zero-viscosity limit. But for $C > 1$, the function $F(z)$ behaves as

$$F(z) \sim z^{-\zeta} \quad (z \rightarrow 0) \tag{1.8}$$

with

$$\zeta = 2\ln C/\ln 2 \quad \text{or} \quad C = 2^{\frac{1}{2}\zeta}, \tag{1.9}$$

so that the spectral behaviour is

$$E(k) \sim k^{-\frac{5}{3}-\zeta} \quad (C > 1) \tag{1.10}$$

in the zero-viscosity limit. Moreover, the energy dissipation ϵ vanishes in the limit $\nu \rightarrow 0$ for $C > 1$. The energy introduced into the system by the external force is not dissipated but, as we shall show later, flows to smaller and smaller wavenumbers $(k < k_0)$.

This reversal in the direction of the flow of energy is not unfamiliar. It has been argued that such a reversal occurs in two-dimensional turbulence (Kraichnan 1967). Behaviour strikingly similar to what we find has been seen by Frisch, Lesieur & Sulem (1976) in a closure-model calculation which permits continuous variation of the spatial dimensionality from $d = 3$ to $d = 2$. As d decreases, an abrupt change in the direction of energy flow occurs, at $d = d_c \approx 2{\cdot}05$, with an accompanying steepening of the energy spectrum power law as in (1.10), until a k^{-3} spectrum is reached at $d = 2$. However, Fournier & Frisch (1978) have examined the continuous-d closure model in more detail and found that the situation near $d = d_c$ is more complicated than a naïve one-to-one relation between our model parameter C and the spatial dimension would suggest. In particular, the closure model, in addition to the Kolmogorov spectrum $k^{-\frac{5}{3}}$ and the equipartition spectrum k^{d-1}, allows *two* stationary power-law-type energy spectra for $d < d_c' \approx 2{\cdot}06$ and *none* for $d > d_c'$, whereas our model allows an anomalous stationary power-law spectrum for both $C < 1$ and $C > 1$.

Generalizations of the cascade-model equations are, of course, easily imagined if one broadens the range of interactions one is willing to allow. Gledzer (1973), for example, has written equations which include couplings to second-nearest-neighbour wavenumber shells (e.g. terms of the type $\dot{u}_n = \text{constant} \times u_{n+1} u_{n+2} + \ldots$), which, with a suitable choice of coefficients, permits conservation of both energy and enstrophy in the inviscid limit. One may likewise relax the requirement of quadratic interaction terms, or increase the number of modes per shell. Fournier & Frisch (1978) have suggested that some such modification might yield a model with a structure more like that of their closure model near $d = 2$, which could be useful in clarifying the stability of the various stationary spectra near $d = d_c$, since direct numerical investigation of the closure model is rather difficult in this region.

We shall nevertheless confine ourselves to the model equations (1.4). These have the advantage of depending in an important way only on the single parameter C (α may be absorbed as a scale factor by the u_n) and yet of reproducing at least qualitatively many of the features of spectral evolution found in the far more complicated closure models of turbulence.

We shall concentrate here on time-dependent solutions of the cascade model. In particular, the possibility of scale-similar evolution in time, described by

$$E(k,t) = \text{constant} \times t^{-2a} k^{-\frac{5}{3}-s} F[kL(t)], \tag{1.11}$$

$$L(t) = L_0(t/t_0)^y, \quad y = (1-a)/(\tfrac{2}{3} - \tfrac{1}{2}s), \qquad (1.12), (1.13)$$

follows naturally from the model equations, as we shall show further on. The region of validity of the scaling form (1.11) depends on whether or not there is an external force, on how large the viscosity is, and on how long the system has been evolving. It is also necessary to distinguish the cases $C < 1$ and $C > 1$. It will be convenient to summarize below what we have learned about the scaling behaviour of the model, expressed in terms of the energy spectrum $E(k, t)$, since the results are distributed over the next six sections and given there in terms of the less familiar model variables $u_n(t)$.

(i) $C < 1$ and all $\mathscr{F}_n = 0$ (no external force). The energy spectrum behaves as

$$E(k,t) \sim [\epsilon(t)]^{\frac{2}{3}} k^{-\frac{5}{3}} F_{\text{I}}[kL(t)] \quad (k \ll K_d(t)), \tag{1.14}$$

$$L(t) = L_0(t/t_0)^y, \tag{1.15}$$

with
$$y = \tfrac{3}{2}(1-a) \quad (\tfrac{1}{3} < a \leqslant 1), \tag{1.16}$$
$$\epsilon(t) \equiv -dE(t)/dt, \tag{1.17}$$
$$\epsilon(t) \sim t^{-3a} \tag{1.18}$$
and
$$F_{\mathrm{I}}(0) = 0, \quad F_{\mathrm{I}}(\infty) = 1. \tag{1.19}$$

The constants L_0 and t_0 depend on the initial conditions, as does the origin of the time variable. The function $F_{\mathrm{I}}(z)$ and the exponents a and y depend on the choice of the parameter C in the model equations (1.14). The scaling form (1.14) is in general valid only for values of k where the effects of viscosity can be ignored ($k \ll K_d$). However, for one particular value of C ($C \simeq 0{\cdot}6$), the exponents take the values
$$a = \tfrac{2}{3}, \quad y = \tfrac{1}{2}, \tag{1.20}$$
and the scaling law applies to the *entire* spectrum.

It has long been suggested that the decay of three-dimensional homogeneous isotropic turbulence proceeds in a self-similar way. The two-volume survey by Monin & Yaglom (1975) contains a thorough discussion of this hypothesis. Experimental evidence from wind-tunnel experiments is inconclusive, since it is not clear whether turbulent energy in the largest scales (smallest wavenumbers) is nearly enough isotropic and undistorted by the finite size of the wind tunnel to permit an adequate test of the idea. The power-law decay of the energy predicted by (1.14),
$$E(t) \sim t^{-w}, \quad w = 3a-1, \tag{1.21), (1.22}$$
is clearly observed, but the value of w seems to depend on the method of generating the turbulence. For instance, values of w ranging from $1{\cdot}0$ to $1{\cdot}3$ have been found by Gad-el-Hak & Corrsin (1974). The longitudinal integral scale
$$L_p(t) \equiv [E(t)]^{-1} \int_0^\infty k^{-1} E(k,t)\, dk, \tag{1.23}$$
which characterizes the scales where most of the energy resides and which is predicted to grow as t^y by (1.15), is seen to increase in the wind-tunnel experiments, but not always according to a simple power law.

The model equations of Heisenberg (1948) and Kovasznay (1948) and the diffusion approximation of Leith (1967) for the evolution of the energy spectrum all permit scaling behaviour as in (1.14). Lesieur & Schertzer (1977) have shown numerically that a Markovian closure model exhibits this scaling behaviour. With initial conditions similar to those studied here they saw an exponent $w = 1{\cdot}33$. But they, like Leith, pointed out that the value of the exponent w depends on the small k behaviour of the initial energy spectrum.

(ii) $1 \leqslant C < 2^{\frac{2}{3}}$ and all $\mathscr{F}_n = 0$ (no external forcing). No energy is dissipated in the limit of zero viscosity. The energy spectrum behaves (for $\nu \rightarrow 0$) as
$$E(k,t) \sim t^{-\frac{2}{3}-\zeta} k^{-\frac{5}{3}-\zeta} F_{\mathrm{II}}[kL(t)], \tag{1.24}$$
where ζ is given by (1.9),
$$L(t) = L_0\, t/t_0 \tag{1.25}$$
and
$$F_{\mathrm{II}}(0) = 0, \quad F_{\mathrm{II}}(\infty) = 1. \tag{1.26}$$

Equation (1.24) implies the homogeneity relation

$$E(\lambda^{-1}k, \lambda t) = \lambda E(k,t). \tag{1.27}$$

It is tempting to set $\zeta = \frac{4}{3}$ in order to try to describe two-dimensional turbulence (in spite of the fact that (1.4) does not conserve mean-squared vorticity!). Indeed, such a scaling law as (1.24) for freely evolving two-dimensional turbulence has been proposed by Batchelor (1969), and there is qualitative agreement between the relation (1.27) and the graphical results of a calculation based on the eddy-damped quasi-normal Markovian approximation given by Pouquet *et al.* (1975). But the equation for $F_{\mathrm{II}}(z)$ which we shall describe later may cease to allow simple power-law scaling relations at precisely the value of C for which $\zeta = \frac{4}{3}$.

(iii) $C > 1$ and $\mathscr{F}_0 \neq 0$ (forcing in a region of wavenumbers around k_0). The energy spectrum for $k > k_0$ has the form

$$E(k,t) \sim k^{-\frac{5}{3}-\zeta}$$

in the absence of viscosity and there is no transfer of energy to large wavenumbers. With viscosity, the spectrum may be written as in (1.6) and (1.8). For $k < k_0$, the spectrum behaves as

$$E(k,t) \sim k^{-\frac{5}{3}} F_{\mathrm{III}}[kL(t)], \tag{1.28}$$

with

$$L(t) = L_0(t/t_0)^{\frac{3}{2}} \tag{1.29}$$

and

$$F_{\mathrm{III}}(0) = 0, \quad F_{\mathrm{III}}(\infty) = 1, \tag{1.30}$$

and (for $\nu = 0$) all the energy introduced by the force flows to small k. There is nothing to prevent us from choosing $C = 2^{\frac{2}{3}}$, so that $\zeta = \frac{4}{3}$. The spectrum then looks very much like that described by Kraichnan (1967), with a k^{-3} behaviour for large k and generation of a $k^{-\frac{5}{3}}$ spectrum as the energy flows to smaller and smaller k. There are no logarithmic corrections to the k^{-3} power law, as suggested by Kraichnan (1971*b*) for two-dimensional turbulence, since the model does not include the effects of non-local transfer of energy.

(iv) $C < 1$ and $\mathscr{F}_0 \neq 0$ (forcing near wavenumber k_0). The energy spectrum for $k > k_0$ is just the stationary one described in (1.6). But for $k < k_0$, a new power law appears, described by

$$E(k,t) \sim k^{-\frac{5}{3}+|\zeta|} F_{\mathrm{IV}}[kL(t)], \tag{1.31}$$

$$L(t) = L_0(t/t_0)^y, \quad y = (\tfrac{2}{3} + \tfrac{1}{2}|\zeta|)^{-1}, \tag{1.32), (1.33}$$

$$F_{\mathrm{IV}}(0) = 0, \quad F_{\mathrm{IV}}(\infty) = 1, \tag{1.34}$$

where $\zeta < 0$ is fixed by (1.9). The rate ϵ_{back} at which energy flows 'backwards' to small k, defined as

$$\epsilon_{\mathrm{back}} \equiv \epsilon_{\mathrm{input}} - \epsilon_{\mathrm{diss}}, \tag{1.35}$$

where $\epsilon_{\mathrm{input}}$ is the rate at which energy is introduced by the external force and ϵ_{diss} is the rate of energy dissipation by viscosity, decreases as time goes on:

$$\epsilon_{\mathrm{back}}(t) \sim [L(t)]^{\frac{3}{2}|\zeta|} \sim t^{-p}, \quad p = \tfrac{3}{2}|\zeta| y. \tag{1.36), (1.37}$$

Note that the rate of energy flow to large k remains essentially constant and equal to $\epsilon_{\mathrm{input}}$.

Energy is transferred to large k along the $k^{-\frac{5}{3}}$ portion of the spectrum. The second power law is not accompanied by any energy transfer. One might try to identify it with the equipartition spectrum $E(k) \sim k^{d-1}$, where d is the dimension of space. This is discussed in § 8.

(v) Following a suggestion of Uriel Frisch's (private communication), we have looked for solutions to the inviscid model equations describing the 'catastrophe' which is believed to occur in a finite time t_* (Brissaud *et al.* 1973). We find that the model equations permit a solution of the form (expressed in terms of the energy spectrum)

$$E(k,t) \sim k^{-\frac{5}{3}-s} F_{\mathrm{V}}[k/K_c(t_* - t)], \tag{1.38}$$

with

$$K_c(\Delta t) \sim (\Delta t)^{-q}, \quad q = (\tfrac{2}{3} - \tfrac{1}{2}s)^{-1}, \qquad (1.39), (1.40)$$

$$F_{\mathrm{V}}(0) = 1, \quad F_{\mathrm{V}}(\infty) = 0, \tag{1.41}$$

where $F_{\mathrm{V}}(z)$ decreases exponentially fast in the limit $z \to \infty$. Note that the results in the previous four cases were all for $t \gg t_*$.

The form (1.38) implies that the enstrophy

$$\Omega = \int_0^\infty k^2 E(k,t)\, dk \tag{1.42}$$

diverges in a time t_* as

$$\Omega(t) \sim (t_* - t)^{-2}. \tag{1.43}$$

It is likely that as $s \to \frac{4}{3}$, $t_* \to \infty$.

We have outlined above how the model behaves in various interesting cases. It has the advantage, over models which treat the energy spectrum in a more sophisticated way, that analytic insight into its properties is relatively easy to obtain. Moreover, it is far easier to perform numerical calculations with the cascade model, which comes with a ready-made logarithmic discretization of the k axis, than with models which retain a continuous wavenumber.

Its most obvious defect is its complete neglect of non-local energy transfer. But curiously enough, the fact that it couples shells in wavenumber space of finite thickness, rather than the infinitesimally thin shells which occur in the completely local differential equation for the energy spectrum of Kovasznay (1948) and the diffusion approximation of Leith (1967), seems to make it behave more realistically in the dissipation region: the energy spectrum falls off exponentially for large k according to the cascade model (Bell & Nelkin 1977), but drops to zero identically beyond a certain multiple of the dissipation wavenumber K_d, according to the differential equations. This difference is especially remarkable, since in the limit of zero shell thickness the cascade-model equations approach the Kovasznay equation and the Leith equation, depending on the choice of C, as we show in the appendix.

In §§ 2–7, the equations describing the scaling behaviour of the model are derived for each of the cases discussed above. Examples from direct numerical integration of the model equations (1.4) are given to illustrate the conclusions reached on the basis of the scaling equations. The results are discussed in § 8.

2. Scaling behaviour of the cascade model

The dynamical equations of the cascade model have been written down in (1.4). In order to establish the possibility of scale-similar evolution of the model in time, it is convenient to introduce a new set of variables which incorporate some of the expected power laws:

$$u_n(t) \equiv A(t/t_0)^{-a} k_n^{-\frac{1}{3}} (k_n/k_0)^{-\frac{1}{2}s} g_n(t). \tag{2.1}$$

The hope is that by factoring out power laws in k and t as in (2.1), we may choose as boundary conditions for the $g_n(t)$

$$\lim_{t\to\infty} g_n(t) = 1. \tag{2.2a}$$

We shall also assume

$$g_n(0) = 0 \tag{2.2b}$$

for all n except $n = 0$, i.e. energy will be assumed to be present initially in the $n = 0$ shell ($u_0(0) \neq 0$).

Substitution of (2.1) into (1.4) yields the following equation for $g_n(t)$:

$$(2^{\frac{2}{3}}\alpha A)^{-1} k_n^{-\frac{2}{3}} (k_n/k_0)^{\frac{1}{2}s} (t/t_0)^a \left[\frac{dg_n}{dt} - \frac{a}{t} g_n\right] = 2^s g_{n-1}^2 - 2^{-\frac{1}{2}s} g_n g_{n+1}$$
$$- C(2^{\frac{1}{2}s} g_{n-1} g_n - 2^{-s} g_{n+1}^2) - \nu (2^{\frac{2}{3}}\alpha A)^{-1} (t/t_0)^a k_n^{\frac{4}{3}} (k_n/k_0)^{\frac{1}{2}s} g_n. \tag{2.3}$$

Since we shall consider only values of n far from $n = 0$ when there is an external force, the forcing term has been omitted in (2.3).

Next the explicit dependence of the left-hand side of (2.3) on k_n is removed by introducing a new time variable

$$\tau \equiv Q k_n t^y, \tag{2.4}$$

with

$$Q = (2^{\frac{2}{3}}\alpha A k_0^{\frac{1}{2}s} t_0^a)^{1/(\frac{2}{3}-\frac{1}{2}s)}, \tag{2.5}$$

$$y = (1-a)/(\tfrac{2}{3} - \tfrac{1}{2}s), \tag{2.6}$$

and new functions

$$\gamma_n(\tau) \equiv g_n(t). \tag{2.7}$$

Equation (2.3) becomes

$$\tau^{-(\frac{2}{3}-\frac{1}{2}s)}[y\tau d\gamma_n(\tau)/d\tau - a\gamma_n(\tau)] = 2^s \gamma_{n-1}^2(\tfrac{1}{2}\tau) - 2^{-\frac{1}{2}s}\gamma_n(\tau)\gamma_{n+1}(2\tau)$$
$$- C[2^{\frac{1}{2}s}\gamma_{n-1}(\tfrac{1}{2}\tau)\gamma_n(\tau) - 2^{-s}\gamma_{n+1}^2(2\tau)] - \nu Q^{-1/y} k_n^{2-1/y} \tau^{a/y} \gamma_n(\tau). \tag{2.8}$$

In the region where the viscosity term may be neglected, there is nothing in the equations to distinguish the behaviour of one $\gamma_n(\tau)$ from another $\gamma_{n'}(\tau)$. We therefore seek solutions to the equations for a single function $\gamma(\tau)$ independent of n:

$$\gamma_n(\tau) = \gamma(\tau), \tag{2.9}$$

$$\tau^{-(\frac{2}{3}-\frac{1}{2}s)}[y\tau d\gamma(\tau)/d\tau - a\gamma(\tau)] = 2^s\gamma^2(\tfrac{1}{2}\tau) - 2^{-\frac{1}{2}s}\gamma(\tau)\gamma(2\tau) - C[2^{\frac{1}{2}s}\gamma(\tfrac{1}{2}\tau)\gamma(\tau) - 2^{-s}\gamma^2(2\tau)], \tag{2.10}$$

with the boundary conditions, suggested by (2.2),

$$\gamma(0) = 0, \quad \gamma(\infty) = 1. \tag{2.11}$$

The boundary conditions for the $g_n(t)$ at $t = 0$ would in fact serve to distinguish the $\gamma_n(\tau)$ from each other at $t = 0$, but one hopes that as t increases the $g_n(t)$ will approach values governed by the single function $\gamma(\tau)$. This can be verified only by stability analysis of the equations or by direct numerical integration. We shall provide evidence of the latter sort.

If solutions of (2.10) and (2.11) can be found, then the above steps imply for the original variables $u_n(t)$ the possibility of scale-similar behaviour in time:

$$u_n(t) \sim t^{-a} k_n^{-\frac{1}{3}-\frac{1}{2}s} \gamma(\tau). \tag{2.12}$$

This, combined with the rough equivalence $\frac{1}{2}u_n^2(t) \sim k_n E(k_n)$, which follows from the definition (1.1), leads directly to the scale-similar behaviour of the energy spectrum given in (1.11). A more formal demonstration of (1.11) may be constructed using the exact definition (1.1).

It is easy to see that (2.10) can have solutions satisfying the boundary conditions (2.11) for only two values of s:

$$s = 0 \quad \text{or} \quad s = \zeta = 2\ln C/\ln 2. \tag{2.13}$$

A normalization condition for $\gamma(\tau)$ closely related to the energy-conserving property of the original equations (1.4) follows from (2.10) and (2.11). Let

$$\tau_n = 2^n \tau_0 \tag{2.14}$$

in (2.10), multiply by $\tau_n^{-\frac{3}{2}}\gamma(\tau)$ and sum over n up to $n = N$. The left-hand side then becomes

$$\sum_{n=-\infty}^{N} \tau_n^{-Z} \gamma(\tau_n)\left[y\tau_n\, d\gamma(\tau_n)/d\tau_n - a\gamma(\tau_n)\right] = \tau_0^{-Z}\left[\tfrac{1}{2}y\tau_0\, d\Gamma_N(\tau_0)/d\tau_0 - a\Gamma_N(\tau_0)\right], \tag{2.15}$$

where we have defined

$$Z \equiv \tfrac{2}{3} + s \tag{2.16}$$

and

$$\Gamma_N(\tau_0) \equiv \sum_{n=-\infty}^{N} 2^{-nZ}\gamma^2(2^n\tau_0). \tag{2.17}$$

After the same operation on the right-hand side of (2.10), we obtain

$$\tau_0^{-Z}\left[\tfrac{1}{2}y\tau_0\, d\Gamma_N(\tau_0)/d\tau_0 - a\Gamma_N(\tau_0)\right] = \tau_N^{-\frac{3}{2}s}\left[2^{-s}C\gamma(2\tau_N) - 2^{-\frac{1}{2}s}\gamma(\tau_N)\right]\gamma(\tau_N)\,\gamma(2\tau_N). \tag{2.18}$$

In nearly all cases we may safely take the limit $N \to \infty$ and use the boundary condition $\gamma(\infty) = 1$ to simplify considerably the right-hand side of (2.18). The nature of the solutions to (2.10) and (2.18), and their implications for the behaviour of the model, are most conveniently treated separately for each of the situations discussed in the introduction.

3. Scaling behaviour for $0 \leqslant C < 1$ and no forcing

Freely evolving three-dimensional homogeneous isotropic turbulence appears to be described by this case. The scale-similar evolution of the variables $u_n(t)$ is given by (2.1) with $s = 0$:

$$u_n(t) = A(t/t_0)^{-a} k_n^{-\frac{1}{3}} \gamma(\tau), \tag{3.1}$$

where

$$\tau = (At_0^a)^{\frac{3}{2}} k_n t^y, \quad y = \tfrac{3}{2}(1-a) \tag{3.2), (3.3}$$

and $\gamma(\tau)$ satisfies the equation

$$\tau^{-\frac{2}{3}}[y\tau d\gamma(\tau)/d\tau - a\gamma(\tau)] = \gamma^2(\tfrac{1}{2}\tau) - \gamma(\tau)\,\gamma(2\tau) - C[\gamma(\tfrac{1}{2}\tau)\,\gamma(\tau) - \gamma^2(2\tau)] \tag{3.4}$$

with the boundary conditions (2.11). In writing these equations, we have set

$$\alpha = 2^{-\frac{2}{3}} \tag{3.5}$$

in order to simplify their appearance without loss of generality. It is evident from equation (1.4) for $u_n(t)$ that results for different values of α are related to each other by simple scale changes of the $u_n(t)$. This choice (3.5) of α will be used henceforward.

Equation (2.18) for $\Gamma_\infty(\tau_0)$ is easily solved in the present case, and yields the normalization condition

$$\sum_{n=-\infty}^{\infty} 2^{-\frac{2}{3}n}\gamma^2(2^n\tau_0) = 2(1-C)\,(3a-1)^{-1}\,\tau_0^{\frac{2}{3}}. \tag{3.6}$$

From this fact and the scaling form (3.1) it follows that the energy of the system, defined in (1.3), decays as an inverse power of time,

$$E(t) = (At_0^a)^3\,(1-C)\,(3a-1)^{-1}\,t^{-3a+1}, \tag{3.7}$$

as does the rate of energy dissipation,

$$\epsilon(t) = \nu \textstyle\sum_n k_n^2\, u_n^2(t) \tag{3.8}$$

$$= (At_0^a)^3\,(1-C)\,t^{-3a}. \tag{3.9}$$

Solutions to (3.4) consistent with the boundary conditions may be obtained in the limits of very small and very large τ. We find that, for $\tau \ll 1$,

$$\gamma(\tau) \approx (2^{\frac{10}{3}}yq/C)\,\tau^{-\frac{5}{3}}\,e^{-q/\tau}, \tag{3.10}$$

and for $\tau \gg 1$,

$$\gamma(\tau) \approx 1 - D\tau^{-\frac{2}{3}}, \tag{3.11}$$

$$D = a[2^{\frac{5}{3}} - 1 - 2^{-\frac{2}{3}} - C(2^{\frac{2}{3}} + 1 - 2^{\frac{1}{3}})]^{-1}. \tag{3.12}$$

However, the unknown exponent a (and y) and the parameter q which appears in (3.10) depend on the value of C in a way which we have not been able to obtain analytically, and must be determined as a kind of eigenvalue by solving (3.4) subject to the boundary conditions. This has proved to be rather difficult, and we have instead resorted to integrating the original equations (1.4) forward in time numerically for various values of C and extracting the exponent a from the observed power-law decay of the energy.

The case $C = 0$ is an exception. For this value of C, we may set $y = 0$, and the derivative term in (3.4) drops out of the equation. It is then a simple matter to solve the equation numerically. One finds that the scaling function $\gamma(\tau)$ vanishes at all but a discrete set of points $\tau_n = 2^n\tau_0$ $(n = 0, 1, 2, \ldots)$, with

$$\tau_0 \approx 1{\cdot}696, \quad \gamma(\tau) \approx 0{\cdot}512. \tag{3.13}$$

The values of $\gamma(\tau_n)$ are generated from the recursion relation obtained from (3.4),

$$\gamma(2\tau) = \tau^{-\frac{2}{3}} + \gamma^2(\tfrac{1}{2}\tau)/\gamma(\tau), \tag{3.14}$$

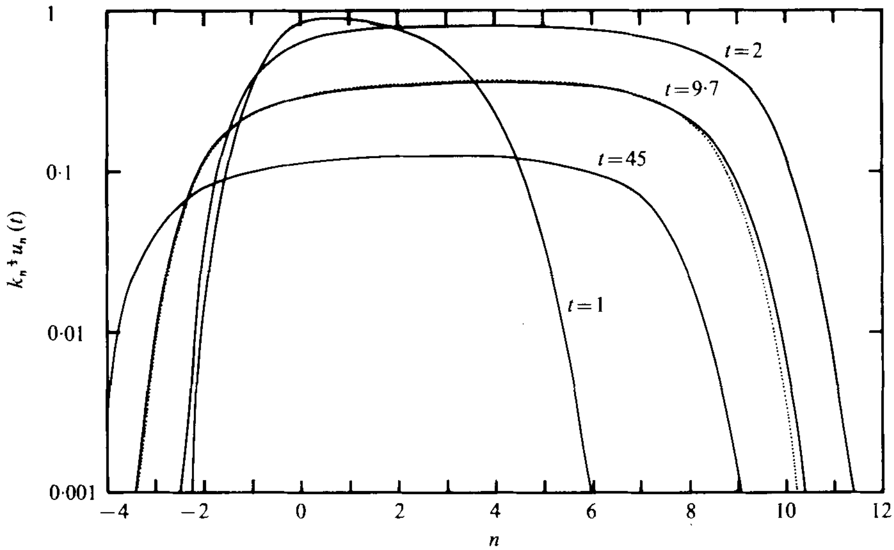

FIGURE 1. Results of numerical integration of the model equations for $C = 0{\cdot}6$, $\nu = 10^{-4}$ ($\alpha = 2^{-\frac{2}{3}}$, $k_0 = 1$). All the energy was located initially in shell $n = 0$, with $u_0(t = 0) = 1{\cdot}3$. The solid lines are drawn to guide the eye; the variables $u_n(t)$ are, of course, defined only for integral n. The dotted curve was generated from the curve $t = 45$ using the scaling relation (3.16) with $a = 0{\cdot}70$.

using as initial values $\gamma(1/2\tau_0) = 0$ and those in (3.13). No other value of τ_0 or $\gamma(\tau_0)$ will generate a sequence of numbers satisfying the boundary conditions (2.11). These results are found to be in excellent agreement with the behaviour of the model for $C = 0$, after a few multiples of the 'eddy-turnover time' $(u_0 k_0)^{-1}$ have elapsed.

An example of the behaviour of the model for $C = 0{\cdot}6$ is shown in figure 1. At $t = 0$ all of the energy was located in the $n = 0$ shell, with $u_0(t = 0) = 1{\cdot}3$. The integration of the equations was carried out using $\alpha = 2^{-\frac{2}{3}}$ and $k_0 = 1$. The quantity

$$h_n(t) \equiv k_n^{\frac{1}{3}} u_n(t) \tag{3.15}$$

has been plotted *vs.* n in order to display better the appearance of the Kolmogorov spectrum, which corresponds in our model to $u_n(t) \sim k_n^{-\frac{1}{3}}$. It can be seen that $h_n(t)$ is level in the region where viscosity is unimportant.

Nearly superimposed on the curve for $t = 9{\cdot}7$ in figure 1 is a dotted curve generated from the curve for $t = 45$ by the scaling law which follows from (3.1)–(3.3):

$$h_{n+p}(2^{-p/y}t) = 2^{ap/y} h_n(t). \tag{3.16}$$

The dotted line was obtained using $p = 1$ and $a = 0{\cdot}70$. The choice for a was extracted from the observed power-law decay of $\epsilon(t)$ using (3.8). The graphs of $\epsilon(t)$ for two values of C in figure 2 show clearly the appearance of the power-law decay.

The scaling behaviour in time of the energy spectrum is evidently established in a reasonably satisfactory way after a few eddy-turnover times $(u_0 k_0)^{-1}$, except in the region where the viscosity term becomes important, where scaling is not expected. The reason that scaling seems to be approximately satisfied even in the dissipation region for the case plotted in figure 1 may be found by returning to (2.8). There it may be seen that for one particular choice of the exponents a and y, mentioned in the introduction in (1.20), the viscosity term ceases to depend on the wavenumber k_n. As

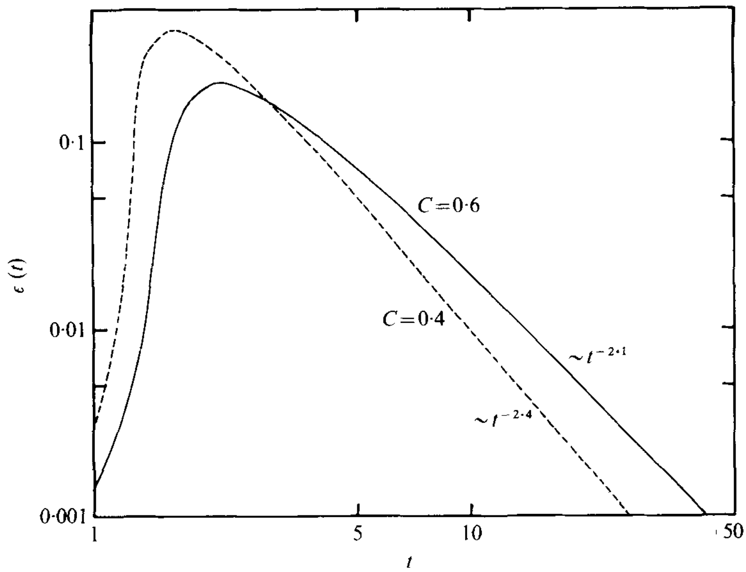

FIGURE 2. Energy dissipation rate $\epsilon(t)$ calculated from (3.8). The solid curve is for the run with $C = 0{\cdot}6$ described in figure 1. The dashed curve is for $C = 0{\cdot}4$ with the values of all other parameters identical to those used for the run with $C = 0{\cdot}6$.

C	w	y
0	2	0
0·2	1·52	0·26
0·4	1·34	0·33
0·6	1·10	0·44
0·8	0·66	0·67
1	0	1

TABLE 1. Numerical results for the exponents w, assuming $E(t) \sim t^{-w}$, and $y = \frac{3}{2}(1-a)$, assuming $\epsilon(t) \sim t^{-3a}$. The viscosity term in (1.4) was replaced by an eddy viscosity for these runs, by truncating the equations at $n = 5$ and setting $u_6(t) = 2^{-\frac{1}{3}}u_5(t)$. The accuracy of the exponents is estimated to be 5 %. The formula $C = 2^{a(1-1/y)}$ provides an approximate fit to the data, but has no theoretical justification.

a consequence, assumption (2.9) becomes possible even in the region where viscosity is important, and (2.10) may be solved with the addition of a term proportional to the viscosity.

Scaling as described by (3.16), valid throughout the spectrum, is thus possible for the value of $C \equiv C_s$ ($C_s \simeq 0{\cdot}6$) for which the exponents a and y are $\frac{2}{3}$ and $\frac{1}{2}$, respectively. The case plotted in figure 1 is very nearly an example of this.

We give in table 1 results for the exponents w and y as a function of the parameter C. The case $C = 0$ has already been investigated by Desnyansky & Novikov (1974*b*), who found $w = 2$. The case $C = 1$ is a special limit where the pure Kolmogorov spectrum transfers no energy, to which we can assign $w = 0$ [see (3.7) and (3.9)]. The other entries in the table are extracted from numerical calculations of the sort described above.

The dimensionless ratio $K_d(t)/L^{-1}(t)$, where K_d is defined in (1.7) and $L(t)$ in (1.12), decreases with time for $C < C_s$, so the region where scale similarity is a useful characterization shrinks with time. For $C > C_s$, on the other hand, the ratio increases, and so, relative to the scale defined by the dissipation wavenumber K_d, scale similarity applies to an increasingly large portion of the spectrum.

4. Scaling behaviour for $1 \leqslant C < 2^{\frac{2}{3}}$ and no forcing

As was mentioned in the introduction, the behaviour of the cascade model for $C > 1$ is similar to what is observed by Frisch *et al.* (1976) for a spatial dimensionality close to 2, in that energy ceases to be dissipated in the zero-viscosity limit and the energy spectrum follows a steeper power law for large k. Unfortunately, as we shall see, the scaling behaviour of the model equations is not of the simple power-law type at precisely the value of C ($C = 2^{\frac{2}{3}}$) which might have described two-dimensional turbulence.

The scaling behaviour for $C \geqslant 1$ and no forcing is obtained by setting s in (2.1) to the value $s = \zeta$ given in (1.9):

$$u_n(t) = A(t/t_0)^{-a} k_n^{-\frac{1}{3}} (k_n/k_0)^{-\frac{1}{2}\zeta}\gamma(\tau). \tag{4.1}$$

The normalization condition given in (2.17) and (2.18) is easy to obtain for this case, since the right-hand side of (2.18) vanishes in the limit $N \to \infty$. The solution of the resulting equation is

$$\Gamma_\infty(\tau_0) \equiv \sum_{n=-\infty}^{\infty} 2^{-(\frac{2}{3}+\zeta)n}\gamma^2(2^n\tau_0) = \Gamma\tau_0^{2a/y}, \tag{4.2}$$

where y is given in (2.6) and Γ is an unknown constant. Since $\Gamma_\infty(\tau_0)$ satisfies, by definition, the identity

$$\Gamma_\infty(2\tau_0) = 2^{\frac{2}{3}+\zeta}\Gamma_\infty(\tau_0),$$

it is necessary that

$$2a/y = \tfrac{2}{3} + \zeta,$$

or, from (2.6),

$$y = 1, \quad a = \tfrac{1}{3} + \tfrac{1}{2}\zeta. \tag{4.3}$$

The scaling variable τ is therefore [see (2.4)]

$$\tau = Qk_n t \tag{4.4}$$

with

$$Q = (Ak_0^{\frac{1}{2}\zeta} t_0^a)^{1/(\frac{2}{3}-\frac{1}{2}\zeta)}, \tag{4.5}$$

and the total energy of the system is

$$E(t) = \tfrac{1}{2}\Gamma[(At_0^a)^2 k_0^\zeta]^{1/(\frac{2}{3}-\frac{1}{2}\zeta)}, \tag{4.6}$$

which is a constant independent of time, as would be expected for a spectrum $E(k) \sim k^{-\frac{5}{3}-\zeta}$, since such a spectrum with $\zeta \neq 0$ does not transfer energy to large wavenumbers in the limit of zero viscosity, as mentioned in the introduction and discussed in detail in Bell & Nelkin (1977). In effect, the exponents given in (4.3) follow from the scaling assumption (4.1) and energy conservation.

The equation for the scaling function $\gamma(\tau)$ appearing in (4.1) is obtained from (2.10):

$$\tau^{-(\frac{2}{3}-\frac{1}{2}\zeta)}[\tau d\gamma(\tau)/d\tau - a\gamma(\tau)] = C^2[\gamma^2(\tfrac{1}{2}\tau) - \gamma(\tfrac{1}{2}\tau)\gamma(\tau)] - C^{-1}[\gamma(\tau)\gamma(2\tau) - \gamma^2(2\tau)] \tag{4.7}$$

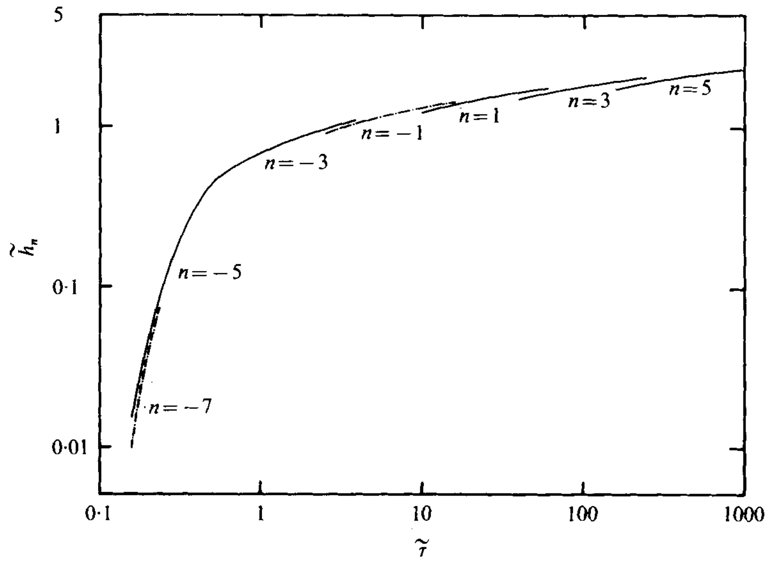

FIGURE 3. The emergence of the scale-similar behaviour predicted in (4.1), for $C = 1{\cdot}5$,

$$\nu = 2{\cdot}6 \times 10^{-7} \quad (\alpha = 1{\cdot}66,\ k_0 = 1)$$

and no forcing. Energy is present initially in shell $n = 0$ with $u_0(t = 0) = 1{\cdot}3$. The quantity $\tilde{h}_n = (k_N t)^{\frac{1}{3}+\frac{1}{2}\zeta} u_n(t)$, which is plotted *vs.* $\tilde{\tau} = k_n t$, should approach a single curve proportional to $\gamma(\tau)$ in (4.1) for each n if the energy spectrum develops in a self-similar way. The exponent $\zeta \simeq 1{\cdot}17$ is given by (1.9). The behaviour of the model is plotted for $5 < t < 30$ only.

with the boundary conditions (2.11). Asymptotic solutions to (4.7) are

$$\gamma(\tau) \approx (2^{\frac{10}{3}} q/C)\, \tau^{-\frac{5}{3}+\frac{1}{2}\zeta} e^{-q/\tau} \quad \text{for} \quad \tau \ll 1 \tag{4.8}$$

and

$$\gamma(\tau) \approx 1 - D\tau^{-(\frac{2}{3}-\frac{1}{2}\zeta)} \quad \text{for} \quad \tau \gg 1 \tag{4.9}$$

with

$$D = 2^{\frac{2}{3}} Ca (2^{\frac{2}{3}} C^2 - 1)^{-1} (2^{\frac{2}{3}} - C)^{-1}. \tag{4.10}$$

Note the persistence of the deviation of $\gamma(\tau)$ from a constant for C near $2^{\frac{2}{3}}$ indicated by (4.9).

In § 3 we verified the scaling behaviour of the system for $C = 0{\cdot}6$ by simply observing that the spectrum at one time was related to the spectrum at a later time by appropriate scale changes. An example of another method is shown in figure 3. The model equations for $C = 1{\cdot}5$ were integrated forward in time up to $t = 30$, starting with all energy initially in shell 0 ($u_0(t = 0) = 1{\cdot}3$). The form given for $u_n(t)$ in (4.1) suggests that if

$$\tilde{h}_n(t) \equiv t^a k_n^{\frac{1}{3}+\frac{1}{2}\zeta} u_n(t)$$

is plotted *vs.* $\tilde{\tau} \equiv k_n t$, a single curve proportional to $\gamma(\tau)$ should emerge. This seems to be the case in figure 3: the points for any given value of n appear to converge to a unique curve with increasing time. Points corresponding to $t < 5$ have been omitted, since about this much time was required for the system to reach a quasi-steady state. Values of n where viscosity was important are also omitted. It is worth remarking that energy was very nearly conserved in spite of the presence of viscosity, whereas for $C < 1$ energy is dissipated at a finite rate no matter how small the viscosity.

The very slow approach of $\tilde{h}_n(\tilde{\tau})$ to a constant for large $\tilde{\tau}$ is explained by the asymptotic behaviour of $\gamma(\tau)$ given in (4.9). For $C = 1{\cdot}5$, one expects $\gamma(\tau) \sim 1-9{\cdot}7\tau^{-0{\cdot}082}$.

5. Scaling behaviour for $C > 1$ and forcing

An external force acting on wavenumbers $k \simeq k_0$, represented by $\mathscr{F}_0 \neq 0$ in the model equation (1.4), is assumed to act on the system for a long time. The small scales reach equilibrium with the force in a time of the order of the eddy-turnover time for scales $\sim 1/k_0$. The spectrum generated for $k \geqslant k_0$ is just the steady-state spectrum

$$u_n(t) = \epsilon_{\text{diss}}^{\frac{1}{3}} k_n^{-\frac{1}{3}} (k_n/k_0)^{-\frac{1}{2}\zeta} f(k_n/K_d) \tag{5.1}$$

discussed extensively in Bell & Nelkin (1977), where ϵ_{diss} is the rate of energy dissipation by viscosity. The essential feature of this spectrum is that, in the limit of zero viscosity, the energy spectrum has the pure power-law behaviour

$$E(k) \sim k^{-\frac{5}{3}-\zeta} \quad (k \geqslant k_0) \tag{5.2}$$

and no energy is transferred to large wavenumbers.

The energy introduced into the system by the external force flows to small wavenumbers, and the spectrum which results for $k < k_0$ is obtained from (2.1) by setting $a = 0$ and $s = 0$:

$$u_n(t) = A k_n^{-\frac{1}{3}} \gamma(\tau) \tag{5.3}$$

(corresponding to a $k^{-\frac{5}{3}}$ energy spectrum) with

$$\tau = A^{\frac{3}{2}} k_n t^{\frac{3}{2}}. \tag{5.4}$$

The scaling function $\gamma(\tau)$ satisfies the equation

$$\tfrac{3}{2}\tau^{\frac{1}{3}} d\gamma(\tau)/d\tau = \gamma^2(\tfrac{1}{2}\tau) - \gamma(\tau)\gamma(2\tau) - C[\gamma(\tfrac{1}{2}\tau)\gamma(\tau) - \gamma^2(2\tau)] \tag{5.5}$$

with the boundary conditions $\gamma(0) = 0$ and $\gamma(\infty) = 1$. This equation has the asymptotic solutions

$$\gamma(\tau) \approx 3(2^{\frac{7}{3}} q C^{-1}) \tau^{-\frac{2}{3}} e^{-q/\tau} \quad \text{for} \quad \tau \ll 1 \tag{5.6}$$

and

$$\gamma(\tau) \approx 1 - D\tau^{-x} \quad \text{for} \quad \tau \gg 1. \tag{5.7}$$

The exponent x is the solution of the equation

$$2^x = (2C-1)/(2-C), \tag{5.8}$$

and the constants q and D must be obtained from the solutions to (5.5).

According to (5.3) and (5.4), the region containing an appreciable fraction of the energy extends down to a wavenumber $(At)^{-\frac{3}{2}}$ which decreases continually with time. The amplitude A in (5.3) may be related to the rate of energy flow to small k by using the normalization condition in (2.18). Equation (2.18) in this case yields the condition

$$\sum_{n=-\infty}^{\infty} 2^{-\frac{2}{3}n} \gamma^2(2^n \tau_0) = 2(C-1)\tau_0^{\frac{2}{3}}. \tag{5.9}$$

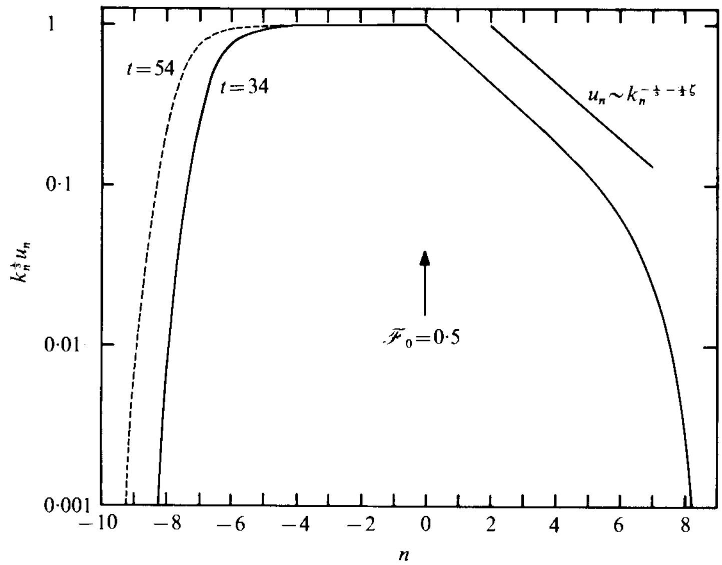

FIGURE 4. Behaviour of the model for $C = 1{\cdot}5$ and an external force $\mathscr{F}_0 = 0{\cdot}5$ chosen according to (5.12) to make $u_0 = 1$. Parameter values were $\nu = 10^{-4}$, $\alpha = 2^{-\frac{2}{3}}$, $k_0 = 1$ and $\zeta \simeq 1{\cdot}17$ [from (1.9)]. Some energy was present initially: $u_0(t = 0) = 1{\cdot}2$. Note the emergence of two power laws $u_n \sim k^{-\frac{1}{3}-\frac{1}{2}\zeta}$ and $k^{-\frac{1}{3}}$ above and below the forcing point respectively.

It follows that the energy of the system increases linearly with time,

$$E(t) = (C-1)A^3t + \text{constant}, \tag{5.10}$$

and that the rate of flow of energy to small k is

$$\epsilon_{\text{back}} = (C-1)A^3. \tag{5.11}$$

By considering the original equations of motion for the variables $u_n(t)$ with $k_n \geqslant k_0$, which are stationary in time, it is not difficult to show that the force $\mathscr{F}_0$ produces a stationary amplitude

$$u_0 = \mathscr{F}_0^{\frac{1}{2}} k_0^{-\frac{1}{2}}(C-1)^{-\frac{1}{2}} \tag{5.12}$$

and

$$\epsilon_{\text{back}} = \mathscr{F}_0^{\frac{3}{2}} k_0^{-\frac{1}{2}}(C-1)^{-\frac{1}{2}}. \tag{5.13}$$

Note that there cannot be a stationary solution with a finite external force for $C = 1$.

There is no problem with setting $C = 2^{\frac{2}{3}}$ when there is forcing (unlike the case without forcing, where the simple power-law scaling relations seem to break down). As remarked in the introduction, the spectrum for this value of C then looks very much like the spectrum for two-dimensional turbulence proposed by Kraichnan (1967), except that the logarithmic corrections expected in two-dimensional flows are absent here, since the model does not include non-local transfer of energy in its equations. It is likely, however, that in the absence of viscosity it would require an infinite amount of time to set up the k^{-3} spectrum up to infinitely large k by a step-by-step cascade of energy, since the turnover times $(u_n k_n)^{-1}$ are the same on all scales k_n.

A graph of the model behaviour for $C = 1{\cdot}5$ is shown in figure 4, with an external force chosen using (5.12) to produce a unit amplitude for u_0. The two power laws above and below k_0 are clearly visible.

6. Scaling behaviour for $C < 1$ and forcing

We assume that the external force $\mathscr{F}_0$ acts on the wavenumber shell $n = 0$. Nearly all of the energy introduced into the system by the force cascades to large k and is dissipated by viscosity at the rate ϵ_{diss}. After a sufficiently long time, of the order of several eddy-turnover times $(u_0 k_0)^{-1}$, a nearly stationary energy spectrum is set up for $k \geqslant k_0$. It is described by the scaling form

$$u_n = \epsilon_{\text{diss}}^{\frac{1}{3}} k_n^{-\frac{1}{3}} f(k_n/K_d) \quad (n \geqslant 0) \tag{6.1}$$

that was discussed in Bell & Nelkin (1977). This corresponds to the usual Kolmogorov $k^{-\frac{5}{3}}$ energy spectrum cut off at the dissipation wavenumber $K_d = \epsilon_{\text{diss}}^{\frac{1}{4}} \nu^{-\frac{3}{4}}$.

The behaviour for $k < k_0$ is unexpected. It is described by (2.1) with $a = 0$ and $s = \zeta$:

$$u_n(t) = A k_n^{-\frac{1}{3}} (k_n/k_0)^{-\frac{1}{2}\zeta} \gamma(\tau) \tag{6.2}$$

with

$$\tau = Q k_n t^y, \quad Q = (A k_0^{\frac{1}{2}\zeta})^y \tag{6.3), (6.4}$$

and

$$y = (\tfrac{2}{3} - \tfrac{1}{2}\zeta)^{-1}. \tag{6.5}$$

A second power law, corresponding to an energy spectrum $E(k) \sim k^{-\frac{5}{3}-\zeta}$ for $k < k_0$, is generated by the external force! Note that ζ, defined in (1.9), is *negative* here, since $C = 2^{\frac{1}{2}\zeta} < 1$.

The scaling function $\gamma(\tau)$ satisfies (2.10) with $a = 0$:

$$y\tau^{\frac{1}{3}+\frac{1}{2}\zeta} d\gamma(\tau)/d\tau = C^2[\gamma^2(\tfrac{1}{2}\tau) - \gamma(\tfrac{1}{2}\tau)\gamma(\tau)] - C^{-1}[\gamma(\tau)\gamma(2\tau) - \gamma^2(2\tau)], \tag{6.6}$$

with the boundary conditions $\gamma(0) = 0$ and $\gamma(\infty) = 1$. The asymptotic solutions to this equation are

$$\gamma(\tau) \approx (2^{\frac{10}{3}} y q C^{-1}) \tau^{-\frac{5}{3}+\frac{1}{2}\zeta} e^{-q/\tau} \quad \text{for} \quad \tau \ll 1 \tag{6.7}$$

and

$$\gamma(\tau) \approx 1 - D\tau^{\frac{3}{2}\zeta} \quad \text{for} \quad \tau \gg 1, \tag{6.8}$$

where the constants q and D must be determined from a complete solution to (6.6) satisfying the boundary conditions. They depend on the value of C alone.

Derivation of the normalization condition which is implied by (2.18) is a bit more complicated for the present case. The complication arises because the energy spectrum $E(k) \sim k^{-\frac{5}{3}-\zeta}$ is *less steep* than the Kolmogorov spectrum, and for $\zeta < -\frac{2}{3}$ the energy content of the spectrum between $k = 0$ and $k = k_0$ is *finite*. Allowance must be made for this in studying the normalization condition.

Substitution of the asymptotic behaviour of $\gamma(\tau)$ given by (6.8) into the right-hand side of (2.18) yields the equation

$$\tfrac{1}{2} y \tau_0^{1-Z} d\Gamma_N(\tau_0)/d\tau_0 = C^{-1}(1 - C^3) D \tag{6.9}$$

with

$$Z = \tfrac{2}{3} + \zeta \tag{6.10}$$

and $\Gamma_N(\tau_0)$ defined by (2.17):

$$\Gamma_N(\tau_0) \equiv \sum_{n=-\infty}^{N} 2^{-nZ} \gamma^2(2^n \tau_0). \tag{6.11}$$

For $Z > 0$ (corresponding to $C > 2^{-\frac{1}{3}} \simeq 0{\cdot}794$) we may take the limit $N \to \infty$ and solve (6.9) to find

$$\sum_{n=-\infty}^{\infty} 2^{-nZ} \gamma^2(2^n \tau_0) = 2(yZC)^{-1}(1 - C^3) D \tau_0^Z. \tag{6.12}$$

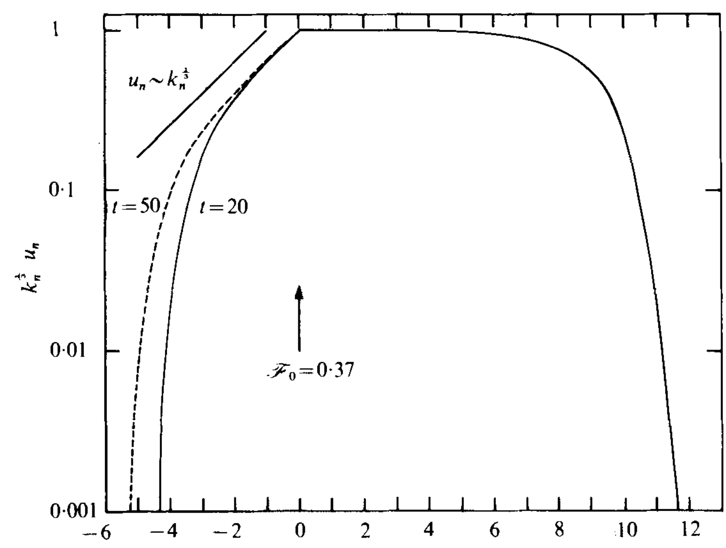

FIGURE 5. Behaviour of the model for $C = 2^{-\frac{2}{3}}$ and an external force $\mathscr{F}_0 = 0{\cdot}37$ chosen according to (6.17) to make $u_0 = 1$. Parameter values were $\nu = 10^{-4}$, $\alpha = 2^{-\frac{2}{3}}$, $k_0 = 1$ and $\zeta = -\frac{4}{3}$ [from (1.9)]. Some energy was present initially: $u_0(t=0) = 1{\cdot}65$. Note the emergence of two power laws $u_n \sim k^{-\frac{1}{3}}$ and $k^{+\frac{1}{3}}$ above and below the forcing point respectively.

But for $Z < 0$ ($C < 2^{-\frac{1}{3}}$), the limit of Γ_N does not exist, and it is necessary to replace $\gamma^2(\tau)$ by $\gamma^2(\tau) - 1$ in the above equations. One then finds, for $C < 2^{-\frac{1}{3}}$, that

$$\sum_{n=-\infty}^{\infty} 2^{-nZ}[\gamma^2(2^n\tau_0) - 1] = 2(yCZ)^{-1}(1-C^3)\,D\tau_0^Z. \tag{6.13}$$

One must work with logarithms for $Z = 0$.

The energy content of the system for $k < k_0$ may be obtained for $Z > 0$ from (6.2) and (6.12). One finds

$$E(t) \sim (yZC)^{-1}(1-C^3)\,DA^{2y}k_0^{y\zeta}\,t^{yZ}. \tag{6.14}$$

The rate at which energy flows to small k therefore decreases as a power of t:

$$\epsilon_{\text{back}}(t) \sim t^{-b} \tag{6.15}$$

with

$$b = 1 - yZ = -\tfrac{3}{2}\zeta(\tfrac{2}{3} - \tfrac{1}{2}\zeta)^{-1}. \tag{6.16}$$

For $Z < 0$, one must use (6.13) instead of (6.12), but one again finds the decline in the accumulation of energy below $k = k_0$ to be described by the power law in (6.15) with the exponent (6.16). Equations (6.15) and (6.16) are valid for all $C < 1$.

Since the rate at which energy flows to small k goes to zero with the passage of time, we may solve the model equations for the amplitude u_0 produced by the force $\mathscr{F}_0$ for large times, assuming that all of the energy flows to large k, and find

$$u_0 = (1-C)^{-\frac{1}{2}}\,k_0^{-\frac{1}{2}}\mathscr{F}_0^{\frac{1}{2}} = Ak_0^{-\frac{1}{3}} \tag{6.17}$$

and

$$\epsilon_{\text{diss}} \to u_0\mathscr{F}_0. \tag{6.18}$$

The results of numerically integrating the model equations for $C = 2^{-\frac{2}{3}} \simeq 0{\cdot}63$ and $\mathscr{F}_0 = 0{\cdot}37$ [chosen to produce a unit amplitude u_0 according to (6.17)] are shown in

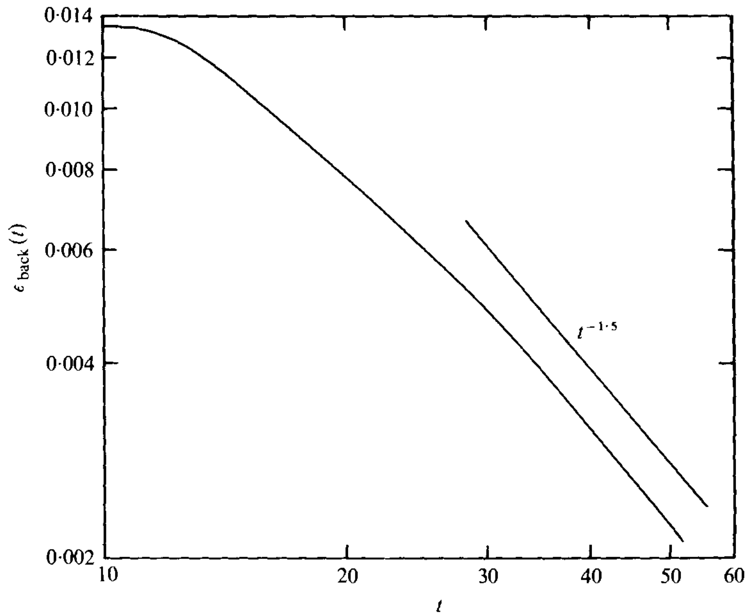

FIGURE 6. The rate ϵ_{back}, which is defined in (6.19) and represents the rate at which energy is stored by the system. Parameters as in figure 5. The power-law fall-off $\sim t^{-1\cdot5}$ is predicted in (6.15) and (6.16). The t^{-1} transient is due to the initial conditions.

figure 5. Both the power laws $u_n \sim k_n^{-\frac{1}{3}}$ for $k > k_0$ and $u_n \sim k_n^{\frac{1}{3}}$ for $k < k_0$ appear. A graph of

$$\epsilon_{\text{back}}(t) \equiv u_0 \mathscr{F}_0 - \epsilon_{\text{diss}} \tag{6.19}$$

is shown in figure 6, and the power law predicted in (6.15) is clearly observed after the passage of a sufficient amount of time, with the exponent $b = 1\cdot5$ predicted by (6.16). The transient t^{-1} behaviour from $t = 15$ to $t = 30$ arises from the decay of the energy present in the system initially.

7. The catastrophe

Brissaud *et al.* (1973) have suggested that solutions to the Euler equations in three dimensions may develop singularities in a finite time (which depends on the initial conditions). The signal for the appearance of singularities is the divergence of the enstrophy [equation (1.42)]. The onset of the 'catastrophe' may be described in terms of the behaviour of the energy spectrum. The energy is supposed to be localized initially around a wavenumber k_0. The energy cascades to larger and larger wavenumbers and a power-law spectrum emerges, extending at time t from k_0 to a cut-off wavenumber $K_c(t)$. The catastrophe arises because this cut-off reaches infinity in a finite time t_*.

Uriel Frisch (private communication) has suggested a scale-similar form for the approach to the catastrophe which is easily adapted to the cascade model. It is only necessary to set

$$a = 0 \tag{7.1}$$

in (2.1), since the large scales will not react to the small quantities of energy transferred to large k, and replace equation (2.4) for τ by

$$\tau = Qk_n(t_* - t)^y, \tag{7.2}$$

with

$$Q = (Ak_0^{\frac{1}{2}s})^y, \quad y = (\tfrac{2}{3} - \tfrac{1}{2}s)^{-1}. \qquad (7.3),\ (7.4)$$

The scaling form for $u_n(t)$ is thus

$$u_n(t) = Ak_n^{-\frac{1}{3}}(k_n/k_0)^{-\frac{1}{2}s}\gamma_*(\tau), \tag{7.5}$$

where s depends on C. The scaling function $\gamma_*(\tau)$ satisfies the equation

$$-y\tau^{\frac{1}{3}+\frac{1}{2}s}\,d\gamma_*(\tau)/d\tau = 2^s\gamma_*^2(\tfrac{1}{2}\tau) - 2^{-\frac{1}{2}s}\gamma_*(\tau)\,\gamma_*(2\tau) - C[2^{\frac{1}{2}s}\gamma_*(\tfrac{1}{2}\tau)\,\gamma_*(\tau) - 2^{-s}\gamma_*^2(2\tau)] \tag{7.6}$$

with boundary conditions (the reverse of the usual ones)

$$\gamma_*(0) = 1, \quad \gamma_*(\infty) = 0. \tag{7.7}$$

The asymptotic solution to (7.6) for $\tau \gg 1$ is

$$\gamma_*(\tau) \sim 2^{\frac{2}{3}}yq\tau^{\frac{1}{3}+\frac{1}{2}s}e^{-q\tau}, \tag{7.8}$$

where q is an unknown constant.

Equations (7.2)–(7.5) for the scaling form of $u_n(t)$ imply the scale-similar approach to the catastrophe described in the introduction in (1.38)–(1.41). The quadratic divergence of the enstrophy $\Omega(t)$ predicted in (1.43) has been observed to describe well the behaviour of $\Omega(t)$ for the cascade model with $C = 0{\cdot}2$, up to the point where viscosity terminates the divergence.

The value of t_* is determined by initial conditions and by the value of C. As discussed in § 5, it is likely that t_* becomes unbounded for $C \to 2^{\frac{2}{3}}$, since in this limit the characteristic time of a step in the cascade no longer decreases with k_n.

8. Discussion

We have studied the scaling behaviour of the cascade model for a variety of situations. In the absence of external forcing, and for $C < 1$, the large wavenumber behaviour is of the familiar 1941 Kolmogorov type. The energy-containing range also behaves in a scale-similar way characterized by a power-law increase of the longitudinal integral scale with time and a power-law decay of the total energy. These power laws depend on the value of the parameter C, which determines the rate at which energy is transferred to small k relative to the transfer rate to large k. The observed decay of the total energy in grid turbulence suggests that a value of C in the range 0·3–0·6 is appropriate for freely decaying three-dimensional turbulence. The energy then decays as t^{-w} with the exponent w near 1.

For forced turbulence we find stationary power-law solutions both above and below the forcing wavenumber. One of these solutions is of the $k^{-\frac{5}{3}}$ type and is associated with an energy cascade. When the parameter C is less than 1, this solution applies at large wavenumbers and describes the familiar energy cascade of three-dimensional turbulence. When C is greater than 1, this solution applies at small wavenumbers and resembles the inverse cascade in two dimensions suggested by Kraichnan (1967). The second power-law solution does not correspond to the cascade of any quantity conserved by the model equations.

The power-law solutions are cut off at large k by viscosity in a familiar way. The cut-off at small k is determined by how long the fluid has been stirred. There is a certain correspondence between the effect of viscosity in terminating the power law for large k and the effect of the stirring time in terminating the power law at small k. For the $k^{-\frac{5}{3}}$ solution and $C < 1$, the viscous cut-off is of the familiar Kolmogorov type and leads to a finite rate of energy cascade in the limit of zero viscosity. For the $k^{-\frac{5}{3}}$ solution and $C > 1$, the cut-off occurs at a wavenumber proportional to $t^{-\frac{3}{2}}$, which allows the stored energy to grow linearly with time. In the limit of infinite stirring time the $k^{-\frac{5}{3}}$ range extends to zero wavenumber, and the energy cascade rate is finite.

For the second power-law solution $\sim k^{-\frac{5}{3}-\zeta}$, the rate of energy flow is determined by the deviations from the pure power law. For $C > 1$ this solution applies at large k, is steeper than $k^{-\frac{5}{3}}$, and has a rate of energy dissipation which goes to zero as the viscosity goes to zero. For $C < 1$ this solution applies at small k, is less steep than $k^{-\frac{5}{3}}$, and has a rate of backward energy flow which goes to zero as the stirring time goes to infinity. Again, the power-law dependence of the cut-off wavenumber and of the cascade rate on stirring time for small k is similar to the power-law dependence on viscosity for large k.

The second power law for small k in forced three-dimensional turbulence is a new prediction of the model which merits further study. One small piece of evidence for a second power law appears in the work of Frisch *et al.* (1976). In their graph for forced '2·05-dimensional turbulence', a second power law is shown for small k. It has, in fact, just the equipartition spectral behaviour k^{d-1}, which is a stationary solution of the d-dimensional closure model (Fourner & Frisch 1978).

If we were to choose the value $C = 2^{-\frac{11}{6}} \simeq 0{\cdot}28$, which would generate with forcing an infrared k^2 energy spectrum appropriate to three dimensions, we could refer to table 1 to see how the energy spectrum would behave for freely evolving turbulence with this choice for C. We should predict an energy decay rate proportional to $t^{-1{\cdot}39}$ in three dimensions, which is not unreasonable. But this manner of choosing C to yield the correct equipartition energy spectrum must clearly break down as the spatial dimensionality approaches $d = 2$. In fact, it is quite likely that if the parameter C could be derived from the Navier–Stokes equation, using an averaging procedure of the sort to be discussed further on, its value would depend on whether or not the fluid was subject to external forcing; the value applicable in the one situation would not necessarily be correct in the other.

The behaviour of the scaling function $\gamma(\tau)$, which appears, for example, in (3.1), and whose square is related to the scaling function F_{I} in (1.14), requires some comment. The asymptotic behaviour of $\gamma(\tau)$ for small τ given in (3.10) indicates that the energy spectrum approaches zero faster than any power of k for small k. This results from the local couplings of the cascade model. In real fluids, the energy spectrum can develop a power law in k for $k \rightarrow 0$ owing to non-local transfer of energy. However, it is reasonable to hope that the behaviour of the energy spectrum in the region of k space where most of the energy is located is largely determined by local transfer, and evolves in the scale-similar way suggested by the cascade model. The power-law fall-off in the total energy of turbulent fluids seen both experimentally and in far more sophisticated treatments of the energy spectrum using the test-field model of

Kraichnan (1971*a*) (J. Herring, private communication; see also Newman 1977; Lesieur & Schertzer 1977) seems to support this.

The case $C < 0$ has not been considered in this paper because the behaviour of the model in this regime does not seem to represent a sensible evolution of the energy spectrum. The initial-value problem does not settle down in a few eddy-turnover times to a quasi-steady power-law-type energy spectrum, and the variables $u_n(t)$ vanish from time to time.

Further understanding and extension of the cascade model depend on establishing its connexion with the basic Navier–Stokes equations. It is reasonable to consider an expansion of the Navier–Stokes equations in a mixed k space, r space representation in which k space is divided into spherical shells of the type used here. Within each shell centred on a wavenumber k_n, we should in principle include variables representing 2^{dn} spatial boxes or other appropriately chosen modes. A strong averaging over these modes so as to neglect within-shell fluctuations would lead to a cascade model of essentially the character considered here. This point of view, that our cascade model is a kind of mean-field theory for strong turbulence, has been discussed previously by Bell & Nelkin (1977) and by Siggia (1977). The inclusion of within-shell fluctuations could then allow for the build-up of intermittency. The recent calculations by Siggia (1977) show one way in which this might come about.

To get the parameter C of the mean-field theory, one can proceed phenomenologically, as in the present paper, or one can estimate coupling coefficients in the mixed representation, and then do the appropriate averaging. These approaches appear to be compatible in principle, but there seem to be important differences between the results obtained so far by these two methods. We find that C is definitely between 0 and 1 and probably between 0·3 and 0·6 in three dimensions. Siggia (1977) estimates coupling coefficients in a spatially local cascade model whose dynamical equations are quite similar to the model studied here. The parameter C in his model is negative and rather large. The explanation for this discrepancy probably lies in the interpretations of the model: our approach refers to overall spectral dynamics for the entire fluid, whereas Siggia's emphasizes spatially local aspects of the cascade.

Besides providing basic insight into the turbulent solutions of the Navier–Stokes equations, investigations along these lines could be profitable in other ways. A better understanding of the relationship of the model to the basic equations should provide useful guidance in constructing more elaborate models designed to study phenomena such as corrections due to anisotropy and to intermittency. The relative ease with which the behaviour of such models can be probed mathematically and numerically gives them a decided advantage over more sophisticated approaches, which quickly become unmanageable when one attempts to treat spectral effects in inhomogeneous anisotropic situations.

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Appendix. Differential limit of the cascade model

The model equations have been written for a cascade of energy from shell to shell in wavenumber space with the typical wavenumbers in each shell increasing by factors of 2 at each step. The factor of 2 is suggested by the quadratic nonlinearity in the Navier–Stokes equations, but may easily be varied. Desnyansky & Novikov (1974*a*) have already suggested how to do this.

The factor of 2 is replaced by an arbitrary factor h, and the variables $u_n(t)$ defined anew as

$$\tfrac{1}{2}u^2(k_n, t) \equiv \int_{h^{-\frac{1}{2}}k_n}^{h^{\frac{1}{2}}k_n} E(k,t)\,dk \tag{A 1}$$

with

$$k_n = h^n k_0. \tag{A 2}$$

The equation which follows from the conditions specified in the introduction in deriving (1.4) is now

$$du(k_n)/dt = \alpha k_n\{u^2(h^{-1}k_n) - hu(k_n)\,u(hk_n) - h^{\frac{1}{3}}C[u(h^{-1}k_n)\,u(k_n) - hu^2(hk_n)]\} - \nu k_n^2\,u(k_n). \tag{A 3}$$

To study the limiting form of the equations as the shell spacing goes to zero, define

$$h = e^{\delta} \tag{A 4}$$

with the intention of taking the limit $\delta \rightarrow 0$. The relation (A 1) in this limit approaches

$$\tfrac{1}{2}u^2(k,t) = \delta k E(k,t). \tag{A 5}$$

It is a simple matter to expand (A 3) in a Taylor series in δ. We find that for $C < 1$ the model equations approach a first-order partial differential equation for the energy spectrum:

$$\frac{\partial E(k,t)}{\partial t} = -K\frac{\partial}{\partial k}k^{\frac{5}{2}}E^{\frac{3}{2}}(k,t) - 2\nu k^2 E(k,t) \tag{A 6}$$

with α chosen such that the limit

$$K = \lim_{\delta\rightarrow 0} 2^{\frac{3}{2}}\delta^{\frac{3}{2}}(1-C)\,\alpha \tag{A 7}$$

is finite. Equation (A 6) is identical to the Kovasznay (1948) model equation.

If instead, guided by the choice $C = 2^{\frac{1}{2}\zeta}$ made for $h = 2$, we set

$$C = h^{\frac{1}{2}\zeta}, \tag{A 8}$$

we find a different limiting behaviour of the model equations. It is necessary to expand (A 3) up to second order in δ; the algebra may be somewhat simplified by using the operator identity exp $(\lambda x d/dx) f(x) = f(e^{\lambda}x)$. We find the limiting equation to be the second-order partial differential equation

$$\frac{\partial E(k,t)}{\partial t} = \beta\frac{\partial}{\partial k}k^{1-\frac{3}{2}\zeta}\frac{\partial}{\partial k}k^{\frac{5}{2}+\frac{3}{2}\zeta}E^{\frac{3}{2}}(k,t) - 2\nu k^2 E(k,t) \tag{A 9}$$

with α now chosen such that the limit

$$\beta = \lim_{\delta\rightarrow 0}\tfrac{1}{3}2^{\frac{2}{3}}\delta^{\frac{5}{2}}\alpha \tag{A 10}$$

is finite. Equation (A 9) would be identical to Leith's (1967) diffusion approximation if we were to choose $\zeta = -\frac{11}{3}$, for which value (A 9) has the two stationary power-law solutions $k^{-\frac{5}{3}}$ and k^2. Leith was led to this choice for ζ by requiring that his equation should have as stationary solutions the equipartition spectrum $E(k) \sim k^2$ as well as the Kolmogorov $k^{-\frac{5}{3}}$ spectrum. We have not fixed ζ in this manner because there does not seem to be any reason to expect the cascade model to embrace at one time both equilibrium and far-from-equilibrium situations.

It is interesting to compare the scaling behaviour seen in our model with what is seen for the diffusion approximation. The diffusion-approximation equation was shown by Leith (1967) to permit a scaling form for the energy spectrum like (1.14). But the scaling function $F_I(\tau)$ was assumed to behave according to a power law $F_I(\tau) \sim \tau^p$ for small τ. The scaling function and the power-law decay of the total energy depended on the choice of the exponent p. In the cascade model only a single power-law decay occurs for a given choice of ζ (assuming that all the energy is initially located near some wavenumber k_0), and $F_I(\tau)$ approaches zero exponentially as $\tau \rightarrow 0$.

This suggests to us that the diffusion-approximation equation may have another scaling solution with a function $F_I(\tau)$ which vanishes identically below some *non-zero* value of τ, just as the equation requires the energy spectrum to go to zero at a finite multiple of the dissipation wavenumber K_d, instead of decreasing exponentially for large k as it does in the cascade model. The diffusion approximation would then yield a unique exponent for the power-law decay of the total energy.

REFERENCES

Batchelor, G. K. 1969 *Phys. Fluids Suppl.* **12**, II 233–239.

Bell, T. L. & Nelkin, M. 1977 *Phys. Fluids* **20**, 345–350.

Brissaud, A., Frisch, U., Léorat, J., Lesieur, M., Mazure, A., Pouquet, A., Sadourny, R. & Sulem, P. L. 1973 *Ann. Geophys.* **29**, 539–546.

Desnyansky, V. N. & Novikov, E. A. 1974*a* *Prikl. Mat. Mekh.* **38**, 507–513.

Desnyansky, V. N. & Novikov, E. A. 1974*b* *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana* **10**, 127–136.

Fournier, J.-D. & Frisch, U. 1978 *Phys. Rev.* A **17**, 747–762.

Frisch, U., Lesieur, M. & Sulem, P. L. 1976 *Phys. Rev. Lett.* **37**, 895–897.

Gad-el-Hak, M. & Corrsin, S. 1974 *J. Fluid Mech.* **62**, 115–143.

Gledzer, E. B. 1973 *Dokl. Akad. Nauk SSSR* **209**, 1046–1048. (See also *Sov. Phys. Dokl.* **18**, 216–217.)

Heisenberg, W. 1948 *Z. Phys.* **124**, 628–657.

Kolmogorov, A. N. 1941 *Dokl. Akad. Nauk SSSR* **30**, 301.

Kovasznay, L. G. S. 1948 *J. Aero. Sci.* **15**, 745–753.

Kraichnan, R. H. 1967 *Phys. Fluids*, **10**, 1417–1423.

Kraichnan, R. H. 1971*a* *J. Fluid Mech.* **47**, 513–524.

Kraichnan, R. H. 1971*b* *J. Fluid Mech.* **47**, 525–535.

Leith, C. E. 1967 *Phys. Fluids* **10**, 1409–1416.

Lesieur, M. & Schertzer, D. 1977 Amortissement auto-similaire d'une turbulence à grand nombre de Reynolds. Preprint.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics: Mechanics of Turbulence*, vol. 2. MIT Press.

Newman, G. R. 1977 A test field model study of a passive scalar in isotropic turbulence. Preprint.

Oboukhov, A. M. 1971 *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana* **7**, 471–475.

Pouquet, A., Lesieur, M., André, J. C. & Basdevant, C. 1975 *J. Fluid Mech.* **72**, 305–319.

Rose, H. & Sulem, P. L. 1978 Fully developed turbulence and statistical mechanics. *J. de Physique* (in press).

Siggia, E. D. 1977 *Phys. Rev.* A **15**, 1730–1750.